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I. G. Rosen
Chien-Hua Frank Su

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Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

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AN APPROXIMATION THEORY FOR THE IDENTIFICATION OF LINEAR THERMOELASTIC SYSTEMS[1]

by

I.G. Rosen[2]

and

Chien-Hua Frank Su

Center for Applied Mathematical Sciences
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113 USA

Abstract

An abstract approximation framework and convergence theory for the identification of thermoelastic systems is developed. Starting from an abstract operator formulation consisting of a coupled second order hyperbolic equation of elasticity and first order parabolic equation for heat conduction, well-posedness is established using linear semigroup theory in Hilbert space, and a class of parameter estimation problems is then defined involving mild solutions. The approximation framework is based upon generic Galerkin approximation of the mild solutions, and convergence of solutions of the resulting sequence of approximating finite dimensional parameter identification problems to a solution of the original infinite dimensional inverse problem is established using approximation results for operator semigroups. An example involving the basic equations of one dimensional linear thermoelasticity and a linear spline based scheme is discussed and numerical results indicating how our approach might be used in a study of damping mechanisms in flexible structures are presented.

1. Introduction

In this paper we develop an abstract approximation framework and convergence theory for the identification of abstract linear thermoelastic systems. The inclusion of thermal effects in the dynamics of flexible structures has recently received an increased amount of attention as an area of research as a result of the problem of solar heating of large flexible spacecraft and the effect that it can have on the structure's vibrational modes and the subsequent design of efficient and effective control laws.

The approach we take here is somewhat different than the traditional one usually taken by other authors working in the area of thermoelasticity (see, for example, [4], [5], [9], [13]). Rather than starting with the mathematical formuation of the basic laws of continuum mechanics and thermo-dynamics and then linearizing to arrive at the so called basic equations of linear thermoelasticity, we consider an abstract operator formuation of these basic equations set in appropriately chosen infinite dimensional spaces. More precisely, we consider an abstract second order equation of elasticity coupled with a first order abstract parabolic equation to describe the dynamics of heat conduction. We then rewrite this system as an equivalent abstract first order system set in an appropriately constructed product Hilbert space. We establish well-posedness using results from linear semigroup theory (i.e. the Lumer-Phillips Theorem) and then proceed to define a class of inverse or parameter identification or estimation problems.

Our approximation results are in the spirit of the treatment by Banks and Ito in [1]. We develop an abstract approximation framework via generic Galerkin approximations to mild solutions of the abstract state equations. A sequence of finite dimensional approximating parameter identification problems result. Using an approximation and convergence result from linear semigroup theory (the Trotter-Kato Theorem; stability and resolvent convergence imply semigroup convergence) we are able to establish that under reasonable and readily verifiable assumptions, solutions to the finite

dimensional identification problems, in some sense, approximate a solution to the original infinite dimensional inverse problem.

We note that in a certain sense, the results that we present here subsume the framework developed in [1]. Indeed, Banks and Ito first treat first order abstract parabolic systems and then consider second order systems. The abstract thermoelestic systems of interest to us here consist of one equation of each type appropriately coupled. By zeroing out the coupling (which the framework we develop below can handle) in our treatment, we obtain most of the results for first and second order systems derived separately by Banks and Ito in [1], simultaneously.

In order to demonstrate that in addition to being theoretically sound, our approximation framework is useful, numerically and computationally feasible, and performs well in practice (or at least, simulation) we apply our general approach to the basic equations of linear one dimensional thermoelasticity. This system consists of a one dimensional wave equation describing the longitudinal or axial vibrations of a slender rod, coupled with the heat equation together with appropriate boundary and initial conditions. We illustrate how a spline based scheme can be constructed so as to conform to our framework and indicate how it might be used in a study of dissipation or damping mechanisms in a flexible structure. Numerical results carried out on a Cray XMP/48 supercomputer are presented and discussed.

An outline of the remainder of the paper is as follows. In the second section we define the abstract thermoelastic system, establish its well-posedness, and state precisely the class of identification problems which are of interest to us. The approximation framework and convergence theory are developed in the third section, and the example and numerical studies are discussed in Section 4. Section 5 contains a brief summary of our findings and some concluding remarks.

2. Abstract Linear Thermoelastic Systems

In this section we define an abstract linear thermoelastic system, and demonstrate the existence, uniqueness, and continuous dependence of solutions via linear semigroup theory. We then define an associated identification or inverse problem.

Let $\mathcal{Q}$ be a metric space and let Q be a compact subset of $\mathcal{Q}$. The set Q will be known as the admissible parameter set. For $j = 1, 2$ let $\{H_j, \langle \cdot, \cdot \rangle_j, |\cdot|_j\}$ be real Hilbert spaces and let $\left\{V_j, \|\cdot\|_j\right\}$ be reflexive Banach spaces. We assume that for $j = 1, 2$, V_j is densely and continuously embedded in H_j with $|\varphi|_j \leq \mu_j \|\varphi\|_j, \varphi \in V_j$. We let V_j^* denote the continuous dual of V_j. Then with H_j as the pivot space, we have $V_j \hookrightarrow H_j = H_j^* \hookrightarrow V_j^*$, with H_j densely and continuously embedded in V_j^*. We denote the usual operator norm on V_j^* by $\|\cdot\|_{j*}, j = 1, 2$. In the usual manner, $\langle \cdot, \cdot \rangle_j$ is understood to denote both the inner product on H_j and the duality pairing on $V_j^* \times V_j$, for $j = 1, 2$.

For each $q \in Q$ we consider what we shall refer to as an abstract linear theormoelastic system given by

$$\ddot{u}(t) + C(q)\dot{u}(t) + A_1(q)u(t) + L(q)^* \theta(t) = f(t; q), \quad t > 0 \tag{2.1}$$

$$\dot{\theta}(t) + A_2(q)\theta(t) - L(q)\dot{u}(t) = g(t; q), \quad t > 0, \tag{2.2}$$

$$u(0) = u_0(q), \dot{u}(0) = v_0(q), \theta(0) = \theta_0(q), \tag{2.3}$$

where for each $t > 0, u(t) \in H_1$ and $\theta(t) \in H_2$. We assume that for each $q \in Q, C(q), A_1(q) \in \mathcal{L}(V_1, V_1^*), A_2(q) \in \mathcal{L}(V_2, V_2^*), L(q) \in \mathcal{L}(V_1, V_2^*), u_0(q) \in V_1, v_0(q) \in H_1, \theta_0(q) \in H_2$, and $f(\cdot; q) \in L_1(0, T; H_1), g(\cdot; q) \in L_1(0, T; H_2)$, for some $T > 0$. The operator $L(q)^* \in L(V_2, V_1^*)$ is defined by

$$\langle L(q)^* \psi, \varphi \rangle_1 = \langle L(q)\varphi, \psi \rangle_2, \psi \in V_2, \varphi \in V_1. \tag{2.4}$$

We shall also require the following further assumptions.

(A) (Symmetry) For each $q \in Q$ the operator $A_1(q)$ is symmetric in the sense that

$$\langle A_1(q)\varphi, \psi \rangle_1 = \langle A_1(q)\psi, \varphi \rangle_1, \varphi, \psi \in V_1.$$

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(B) (Uniform Coercivity) For $j = 1, 2$ there exist constants $\alpha_j, \beta_j \in \mathbf{R}$, independent of $q \in Q$, with $\beta_j > 0$ such that

$$\langle A_j(q)\varphi, \varphi \rangle_j + \alpha_j |\varphi|_j^2 \geq \beta_j \|\varphi\|_j^2, \varphi \in V_j.$$

(Note that without loss of generality we may assume that $\alpha_j \geq 0, j = 1, 2$.)

(C) (Uniform Boundedness) There exist positive constants $\gamma_j, j = 1, 2, \delta$, and ρ, independent of $q \in Q$, for which

$$\|A_j(q)\varphi\|_{j*} \leq \gamma_j \|\varphi\|_j, \varphi \in V_j, j = 1, 2$$

$$\|C(q)\varphi\|_{1*} \leq \delta \|\varphi\|_1, \varphi \in V_1,$$

$$\|L(q)\varphi\|_{2*} \leq \rho \|\varphi\|_1, \varphi \in V_1.$$

(Note that the final bound above implies that $\|L(q)^*\psi\|_{1*} \leq \rho \|\psi\|_2, \psi \in V_2$, as well.)

(D) (Nonnegativity) For each $q \in Q$ the operator $C(q)$ is nonnegative in the sense that $\langle C(q)\varphi, \varphi \rangle_1 \geq 0, \varphi \in V_1$.

(E) (Continuity) For $\varphi \in V_1$ and $\psi \in V_2$ the mappings $q \to A_1(q)\varphi, q \to C(q)\varphi, q \to L(q)^*\psi, q \to A_2(q)\psi$, and $q \to L(q)\varphi$ are continuous from $Q \subset \mathcal{Q}$ into V_1^* or V_2^* (which ever is appropriate). Also, the mappings $q \to u_0(q), q \to v_0(q)$, and $q \to \theta_0(q)$ are continuous from $Q \subset \mathcal{Q}$ into V_1, H_1, and H_2, respectively, as are the mappings $q \to f(t;q)$ and $q \to g(t;q)$ into H_1 and H_2, respectively, for almost every $t \in [0, T]$.

(Note that the two "strong" continuity assumptions on the operators $L(q)$ and $L(q)^*$ can be replaced by the single, somewhat stronger continuity assumption that the mapping $q \to L(q)$ is continuous from $Q \subset \mathcal{Q}$ into $\mathcal{L}(V_1, V_2^*)$ - i.e. $q \to L(q)$ is continuous with respect to the uniform operator topology on $\mathcal{L}(V_1, V_2^*)$.)

(F) (Uniform Domination) There exist functions $f_0, g_0 \in L_1(0, T)$, independent of $q \in Q$, for which $|f(t;q)|_1 \leq f_0(t)$ and $|g(t;q)|_2 \leq g_0(t)$, a.e. $t \in [0, T]$ and every $q \in Q$.

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Remark A comment regarding our referring to the system (2.1) - (2.3) as an abstract thermoelastic system would be in order here. If one thinks of (2.1) as an abstract elasticity equation and (2.2) as an abstract heat equation, then by making some rather simple assumptions concerning thermal stress, and applying some basic mechanical and conservation of energy principles from continuum and thermal mechanics, one could justify the form of the coupling between the two equations (i.e. the operators $L(q)$ and $L(q)^*$). Typically however, the basic equations (both linear and nonlinear) of thermoelasticity are formulated from first principles using nontrivial mathematical formulations of the basic laws of mechanics and thermodyamics (see, for example, [4], [9], or [13]). In light of this, we justify our reference to (2.1) - (2.3) as an abstract thermoelastic system by simply stating that the thermo-mechancial systems of interest to us here, and in particular the basic equations of linear thermoelasticity in one and higher dimensions (see [4],[5], and [16]), can, via appropriate identifications, be put in this abstract form.

We demonstrate the well-posedness of the system (2.1) - (2.3) for each $q \in Q$ by first rewriting it as an equivalent first order system in an appropriate product Hilbert space and then applying results from linear semigroup theory.

Let X be the Banach space defined by $X = V_1 \times H_1 \times H_2$ with norm $|\cdot|_X$ given by $|(\varphi, \psi, \eta)|_X = \left(\|\varphi\|_1^2 + |\psi|_1^2 + |\eta|_2^2 \right)^{1/2}$. For each $q \in Q$ let $X(q)$ denote the Hilbert space which is set equivalent to X and which is endowed with the inner product $\langle \cdot, \cdot \rangle_q$ given by

$$\langle (\varphi_1, \psi_1, \eta_1), (\varphi_2, \psi_2, \eta_2) \rangle_q = \langle A_1(q) \varphi_1, \varphi_2 \rangle_1$$

$$+ \alpha_1 \langle \varphi_1, \varphi_2 \rangle_1 + \langle \psi_1, \psi_2 \rangle_1 + \langle \eta_1, \eta_2 \rangle_2,$$

for $(\varphi_i, \psi_i, \eta_i) \in X, i = 1, 2$. We denote the norm on $X(q)$ induced by the inner product $\langle \cdot, \cdot \rangle_q$ by $|\cdot|_q$ and note that it is immediately clear that assumptions (B) and (C) imply that the norms $|\cdot|_X$

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and $|\cdot|_q$ are equivalent, uniformly in q for $q \in Q$. That is, there exist positive constants m and M, independent of $q \in Q$, for which

$$m\,|\cdot|_q \leq |\cdot|_X \leq M\,|\cdot|_q\,. \tag{2.5}$$

For each $q \in Q$ define the operator $A(q) : Dom(A(q)) \subset X(q) \to X(q)$ by

$$A(q) = \begin{bmatrix} 0 & I & 0 \\ -A_1(q) & -C(q) & -L(q)^* \\ 0 & L(q) & -A_2(q) \end{bmatrix}, \tag{2.6}$$

$$Dom(A(q)) = \{(\varphi, \psi, \eta) \in V_1 \times V_1 \times V_2 : A(q)(\varphi, \psi, \eta) \in X\}. \tag{2.7}$$

Theorem 2.1 For each $q \in Q$ the operator $A(q)$ defined in (2.6), (2.7) above is the infinitesimal generator of a $\mathcal{C}_0$-semigroup, $\{S(t;q) : t \geq 0\}$ of bounded linear opearors on $X(q)$ (and therefore X as well).

<u>Proof</u> The result will follow from, for example, Showalter [12], Theorem IV.4.C, if we can show that for some $\omega \in \mathbf{R}$ the operator $A(q) - \omega I$ is dissipative and the operator $\lambda - A(q)$ is surjective for some $\lambda > \omega$. Toward this end, for $x = (\varphi, \psi, \eta) \in Dom(A(q))$ we have

$$\langle A(q)x, x\rangle_q = \langle(\psi, -A_1(q)\varphi - C(q)\psi - L(q)^*\eta, L(q)\psi - A_2(q)\eta), (\varphi, \psi, \eta)\rangle_q$$

$$= \langle A_1(q)\psi, \varphi\rangle_1 + \alpha_1\langle\psi, \varphi\rangle_1 - \langle A_1(q)\varphi, \psi\rangle_1 - \langle C(q)\psi, \psi\rangle_1$$

$$\quad - \langle L(q)^*\eta, \psi\rangle_1 + \langle L(q)\psi, \eta\rangle_2 - \langle A_2(q)\eta, \eta\rangle_2$$

$$= \alpha_1\langle\psi, \varphi\rangle_1 - \langle C(q)\psi, \psi\rangle_1 - \langle A_2(q)\eta, \eta\rangle_2$$

$$\leq \frac{\alpha_1}{2}|\varphi|_1^2 + \frac{\alpha_1}{2}|\psi|_1^2 - \beta_2\|\eta\|_2^2 + \alpha_2|\eta|_2^2$$

$$\leq \frac{\alpha_1\mu_1^2}{2}\|\varphi\|_1^2 + \frac{\alpha_1}{2}|\psi|_1^2 + \alpha_2|\eta|_2^2$$

$$\leq \frac{\alpha_1\mu_1^2}{2\beta_1}\left\{\langle A_1(q)\varphi, \varphi\rangle_1 + \alpha_1|\varphi|_1^2\right\} + \frac{\alpha}{2}|\psi|_1^2 + \alpha_2|\eta|_2^2$$

$$\leq \omega\,|x|_q^2\,,$$

where $\omega = \max\left\{\frac{\alpha_1\mu_1^2}{2\beta_1}, \frac{\alpha_1}{2}, \alpha_2\right\}$. Recall that we assumed, without loss of generality, that $\alpha_1, \alpha_2 \geq 0$, and note that when $\alpha_1 = \alpha_2 = 0$, we have $\omega = 0$ and that the operator $A(q)$ is dissipative.

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In order to eliminate some rather tedious technical details, we assume for the remainder of the proof that $\alpha_1 = \alpha_2 = 0$ (and therefore $\omega = 0$). The arguments to follow remain, in substance, essentially unchanged in the more general case of $\alpha_1, \alpha_2 \geq 0$. Let $\lambda > 0$, let $(\varphi, \psi, \eta) \in X$ and consider

$$(\lambda - A(q))(u, v, \theta) = (\varphi, \psi, \eta);$$

or equivalently

$$\lambda u - v = \varphi \tag{2.8}$$

$$\lambda v + A_1(q) u + C(q) v + L(q)^* \theta = \psi \tag{2.9}$$

$$\lambda \theta - L(q) v + A_2(q) \theta = \eta. \tag{2.10}$$

Solving (2.8) for v and then substituting into (2.9) and (2.10) we obtain

$$\left(\lambda^2 + \lambda C(q) + A_1(q)\right) u + L(q)^* \theta = \psi + (C(q) + \lambda)\varphi \tag{2.11}$$

$$-L(q) u + \left(1 + \lambda^{-1} A_2(q)\right) \theta = \lambda^{-1}(\eta + L(q)\varphi). \tag{2.12}$$

Let $H = H_1 \times H_2$ be endowed with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle \Phi, \Psi \rangle = \langle \Phi_1, \Psi_1 \rangle_1 + \langle \Phi_2, \Psi_2 \rangle_2, \tag{2.13}$$

for $\Phi = (\Phi_1, \Phi_2), \Psi = (\Psi_1, \Psi_2) \in H$, and let $V = V_1 \times V_2$ be endowed with the norm $\|\cdot\|$ given by

$$\|\Phi\| = \left(\|\Phi_1\|_1^2 + \|\Phi_2\|_2^2\right)^{1/2}, \tag{2.14}$$

for $\Phi = (\Phi_1, \Phi_2) \in V$. Then H is a Hilbert space, V is a reflexive Banach space and $V \hookrightarrow H \hookrightarrow V^*$, with the embeddings dense and continuous. Define the operator $A_\lambda(q) \in \mathcal{L}(V, V^*)$ by

$$A_\lambda(q) = \begin{bmatrix} \lambda^2 + \lambda C(q) + A_1(q) & L(q)^* \\ -L(q) & \left(1 + \lambda^{-1} A_2(q)\right) \end{bmatrix}. \tag{2.15}$$

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Then for $\Phi = (\Phi_1, \Phi_2) \in V$, it is easily shown that

$$\langle A_\lambda(q)\Phi, \Phi \rangle \geq \beta_1 \|\Phi_1\|_1^2 + \lambda^{-1}\beta_2 \|\Phi_2\|_2^2$$

$$\geq \beta_\lambda \|\Phi\|,$$

(2.16)

where $\beta_\lambda = \min\{\beta_1, \lambda^{-1}\beta_2\} > 0$. It therefore follows that $A_\lambda(q)$ is an isomorphism from V onto V^* (see, for example, Tanabe [15], Theorem 2.2.2). Recalling (2.8), (2.11) and (2.12) and setting

$$(u, \theta) = A_\lambda(q)^{-1}\left(v + (C(q) + \lambda)\varphi, \lambda^{-1}(\eta + L(q)\varphi)\right)$$

(2.17)

and

$$v = \lambda u - \varphi,$$

(2.18)

we obtain that $\lambda - A(q)$ is a surjection, and the theorem is proved.

For each $q \in Q$ define $x(\cdot; q) \in C([0, T]; X)$ by

$$x(t; q) = S(t; q)x_0(q) + \int_0^t S(t - s; q)F(s; q)\,ds,$$

(2.19)

for $t \in [0, T]$ where $\{S(t; q) : t \geq 0\}$ is the semigroup generated by $A(q)$, $x_0(q) = (u_0(q), v_0(q), \theta_0(q)) \in X$, and $F(t; q) = (0, f(t; q), g(t; q))$, a.e. $t \in [0, T]$. We shall refer to the X-valued function $x(\cdot; q) = (u(\cdot; q), v(\cdot; q), \theta(\cdot; q))$ given by (2.19) above as the unique mild solution to the abstract theormoelastic system (2.1) - (2.3).

We now proceed to define an identification problem corresponding to (2.1) - (2.3). Let Z denote an observation space. For $i = 1, 2, \ldots, \nu$ and $z \in Z$, let $\Phi_i(\cdot, \cdot; z) : X \times Q \to \mathbf{R}^+$ denote a continuous map from $X \times Q \subset X \times Q$ into the nonnegative real numbers. We consider the following parameter estimation or inverse problem.

(ID) Given observations $\{z_i\}_{i=1}^\nu \subset \underset{i=1}{\overset{\nu}{\times}} Z$ at times $\{t_i\}_{i=1}^\nu \subset \underset{i=1}{\overset{\nu}{\times}} [0, T]$ determine parameters $q \in Q$ which minimize

$$J(q) = \sum_{i=1}^\nu \Phi_i(x(t_i; q), q; z_i)$$

(2.20)

where for each $q \in Q$ and $t_i \in [0, T]$, $x(t_i; q)$ is given by (2.19).

We do not address the question of the existence of a solution $\bar{q}$ to problem (ID) here, but rather defer this discussion until the next section. In the section to follow, the existence of a minimizer for the functional J given by (2.20) is established as a consequence of our approximation theory.

3. Approximation and Convergence

Using a standard Galerkin technique we construct a sequence of finite dimensional approximations to the abstract thermoelastic system (2.1) - (2.3). A sequence of approximating identification problems result. Using a well known approximation theoretic result for linear semigroups (i.e. the Trotter-Kato theorem) we establish that a solution to the infinite dimensional parameter estimation problem, (ID), exists, and moreover, that it is in some sense approximated by solutions to the finite dimensional approximating identification problems.

For $j = 1, 2$ and for each $n_j = 1, 2, \dots$ let $H_j^{n_j}$ be a finite dimensional subspace of H_j with $H_j^{n_j} \subset V_j$, for all n_j. Let $P_j^{n_j} : H_j \to H_j^{n_j}$ denote the orthogonal projection of H_j onto $H_j^{n_j}$ computed with respect to the $\langle \cdot, \cdot \rangle_j$ inner product. We shall require the following approximation assumption.

(G) (Approximation) For $j = 1, 2$

$$\lim_{n_j \to \infty} \left\| P_j^{n_j} \varphi - \varphi \right\|_j = 0, \quad \varphi \in V_j.$$

Note that the dense and continuous embedding of V_j in H_j together with assumption (G) also yield that $\lim_{n_j \to \infty} \left| P_j^{n_j} \varphi - \varphi \right|_j = 0$, $\varphi \in H_j, j = 1, 2$.

For each $q \in Q$ we define the operators $A_j^{n_j}(q) \in \mathcal{L}(H_j^{n_j}), j = 1, 2, C^{n_1}(q) \in \mathcal{L}(H_1^{n_1})$, and $L^n(q) \in \mathcal{L}(H_1^{n_1}, H_2^{n_2})$, $n = (n_1, n_2)$, using standard Galerkin approximation. More precisely, for $j = 1, 2$, and $\varphi^{n_j} \in H_j^{n_j}$, we set $A_j^{n_j}(q)\varphi^{n_j} = v^{n_j} \in H_j^{n_j}$ where v^{n_j} is defined via the Riesz

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Representation Theorem on the Hilbert space $H_j^{n_j}$ as $\langle A_j(q)\varphi^{n_j}, \chi^{n_j}\rangle_j = \langle \psi^{n_j}, \chi^{n_j}\rangle_j$, $\chi^{n_j} \in H_j^{n_j}$.

Similarly we set $C^{n_1}(q)\varphi^{n_1} = \psi^{n_1}$ where $\psi^{n_1} \in H_1^{n_1}$ satisfies $\langle C(q)\varphi^{n_1}, \chi^{n_1}\rangle_1 = \langle \psi^{n_1}, \chi^{n_1}\rangle_1$,

$\chi^{n_1} \in H_1^{n_1}$, and $L^n(q)\varphi^{n_1} = \psi^{n_2}$ where $\psi^{n_2} \in H_2^{n_2}$ satisfies $\langle L(q)\varphi^{n_1}, \chi^{n_2}\rangle_2 = \langle \psi^{n_2}, \chi^{n_2}\rangle_2$,

$\chi^{n_2} \in H_2^{n_2}$. We define the operator $L^n(q)^* \in \mathcal{L}(H_2^{n_2}, H_1^{n_1})$ to be the Hilbert space adjoint of the

operator $L^n(q)$.

We set $u_0^{n_1}(q) = P_1^{n_1}u_0(q), v_0^{n_1}(q) = P_1^{n_1}v_0(q)$ and $\theta_0^{n_2}(q) = P_2^{n_2}\theta_0(q)$, and set $f^{n_1}(t;q) =$

$P_1^{n_1}f(t;q)$ and $g^{n_2}(t;q) = P_2^{n_2}g(t;q)$ for almost every $t \in [0,T]$. We then consider the finite

dimensional system of ordinary differential equations in $H^n = H_1^{n_1} \times H_2^{n_2}$ given by

$$\ddot{u}^n(t) + C^{n_1}(q)\dot{u}^n(t) + A_1^{n_1}(q)u^n(t) + L^n(q)^*\theta^n(t)$$
$$= f^{n_1}(t;q), \quad t > 0, \tag{3.1}$$

$$\dot{\theta}^n(t) + A_2^{n_2}(q)\theta^n(t) - L^n(q)\dot{u}^n(t) = g^{n_2}(t;q), \quad t > 0, \tag{3.2}$$

$$u^n(0) = u_0^{n_1}(q), \dot{u}^n(0) = v_0^{n_1}(q), \theta^n(0) = \theta_0^{n_2}(q). \tag{3.3}$$

We next proceed to rewrite (3.1) - (3.3) as an equivalent first order system. For each $n_1, n_2 =$

$1, 2, \ldots$ and $n = (n_1, n_2)$ let $X^n = H_1^{n_1} \times H_1^{n_1} \times H_2^{n_2}$ be considered as a subspace of the Banach

space X, and for each $q \in Q$ let $X^n(q) = X^n$ be considered as a subspace of the Hilbert space

$X(q)$. For each $q \in Q$ let $A^n(q) \in \mathcal{L}(X^n)$ be given by

$$A^n(q) = \begin{bmatrix} 0 & I & 0 \\ -A_1^{n_1}(q) & -C^{n_1}(q) & -L^n(q)^* \\ 0 & L^n(q) & -A_2^{n_2}(q) \end{bmatrix},$$

let $F^n(t;q) = (0, f^{n_1}(t;q), g^{n_2}(t;q))$, a.e. $t \in [0,T]$, and let $x_0^n(q) = (u_0^{n_1}(q), v_0^{n_1}(q), \theta_0^{n_2}(q))$.

Setting $x^n(t) = (u^n(t), \dot{u}^n(t), \theta^n(t))$, we rewrite (3.1) - (3.3) as

$$\dot{x}^n(t) = A^n(q)x^n(t) + F^n(t;q), \quad t > 0 \tag{3.4}$$

$$x^n(0) = x_0^n(q). \tag{3.5}$$

The solution to the initial value problem (3.4), (3.5) is given by

$$x^n(t;q) = S^n(t;q)x_0^n(q) + \int_0^t S^n(t-s;q)F^n(s;q)\,ds \tag{3.6}$$

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for $t \in [0, T]$, where the semigroup of bounded linear operators on X^n (or $X^n(q)$), $\{S^n(t; q) : t \geq 0\}$, is given by

$$S^n(t; q) = \exp(t A^n(q)), \quad t \geq 0.$$

We define a sequence of finite dimensional approximating identification problems as follows.

(ID^n) Given observations $\{z_i\}_{i=1}^{\nu} \subset \underset{i=1}{\overset{\nu}{\times}} Z$ at times $\{t_i\}_{i=1}^{\nu} \subset \underset{i=1}{\overset{\nu}{\times}} [0, T]$, determine parameters $\bar{q}^n \in Q$ which minimize

$$J^n(q) = \sum_{i=1}^{\nu} \Phi_i(x^n(t_i; q), q; z_i)$$

where for each $q \in Q$ and $t_i \in [0, T]$, $x^n(t_i; q)$ is given by (3.6).

The type or kind of convergence results that we are about to summarize and discuss here have been presented in detail in a number of places in the literature in the context of a variety of identification problems for various types of distributed parameter systems (see, for example [1] and [2]). In the context of the problems (ID) and (ID^n) defined above, our general convergence framework and theory takes the following form.

Using standard continuous dependence results for ordinary differential equations (see, for example, [8], Theorem I.3.2) it is easily argued that the map $q \to x^n(t; q)$ is continuous from $Q \subset \mathcal{Q}$ into X^n for each $t \in [0, T]$, and each $n \in \mathbf{Z}^+ \times \mathbf{Z}^+$. It follows therefore, that the map $q \to J^n(q)$ is continuous for each n. Consequently, since Q has been assumed to be a compact subset of $\mathcal{Q}$, the existence of a solution $\bar{q}^n$ to problem (ID^n) for each n is assured.

Form a directed set from $\mathbf{Z}^+ \times \mathbf{Z}^+$ in the canonical way. That is, for $m = (m_1, m_2)$, $n = (n_1, n_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+$, we say $m \leq n$ if and only if $m_1 \leq n_1$ and $m_2 \leq n_2$. Consider next the following proposition.

<u>Proposition 3.1</u> If $\{q^n\}$ is a net from the directed set $\mathbf{Z}^+ \times \mathbf{Z}^+$ into $Q \subset \mathcal{Q}$ with $q^n \to q$, then $x^n(t; q^n) \to x(t; q)$ for each $t \in [0, T]$ where x^n and x are given by (3.6) and (2.19), respectively.

It is straight forward to argue (see [2], Chapter III, Section 1) that if Proposition 3.1 can be verified, then the fact that Q is a compact subset of $\mathcal{Q}$ implies that the net $\{\bar{q}^n\}$ formed from the solutions to the problems (ID^n), admits a convergent subnet, $\{\bar{q}^m\}$, whose limit, $\{\bar{q}\}$, is a solution to Problem (ID). Thus if Proposition 3.1 can be verified, we have established the existence of a solution to problem (ID), and the fact that it is in some sense approximated by the solutions to the finite dimensional identification problems (ID^n).

Since Galerkin approximation was used to construct the operators $A_j^{n_j}(q), j = 1, 2, C^{n_1}(q)$, and $L^n(q)$, the same arguments used to establish that the operator $A(q) - \omega I$ is dissipative in the proof of Theorem 2.1, yield that the operators $A^n(q) - \omega I$ are dissipative, for all $n \in \mathbf{Z}^+ \times \mathbf{Z}^+$ and $q \in Q$ with ω independent of n and q. It follows, therefore, that the semigroups $\{S^n(t; q) : t \geq 0\}$ are uniformly exponentially bounded, uniformly in n and q. That is,

$$|S^n(t; q)|_q \leq e^{\omega t}, \quad t \geq 0,$$

or, recalling (2.5), that

$$|S^n(t; q)|_X \leq \frac{M}{m} e^{\omega t}, \quad t \geq 0.$$

Thus, using assumptions (E) - (G), Proposition 3.1 will follow immediately if we can show that

$$\lim_{n \to \infty} |S^n(t; q^n) P^n x - S(t; q) x|_X = 0 \tag{3.7}$$

for each $x \in X$ and $t \in [0, T]$ whenever $q^n \to q$, where the projection like mapping $P^n : X \to X^n$ is given by $P^n(\varphi, \psi, \eta) = (P_1^{n_1}\varphi, P_1^{n_1}\psi, P_2^{n_2}\eta)$, for $(\varphi, \psi, \eta) \in X$.

We establish (3.7) using a version of the well-known Trotter-Kato semigroup approximation theorem given in [2] (Theorem II.1.14 on page 40). According to this theorem, the convergence in (3.7) will follow once we have shown that for some $\lambda > \omega$

$$\lim_{n \to \infty} \left| (\lambda - A^n(q^n))^{-1} P^n x - P^n (\lambda - A(q))^{-1} x \right|_{q^n} = 0 \qquad (3.8)$$

for each $x \in X$ wherever $q^n \to q$.

In light of (2.5) and assumption (G), the convergence stated in (3.8) will follow if we can argue that

$$\lim_{n \to \infty} \left| (\lambda - A^n(q^n))^{-1} P^n x - (\lambda - A(q))^{-1} x \right|_X = 0 \qquad (3.9)$$

for each $x \in X$ and some $\lambda > \omega$, wherever $q^n \to q$. For simplicity, we again assume that $\alpha_1 = \alpha_2 = 0$ and therefore that $\omega = 0$. The proof in the more general case when $\alpha_1 = \alpha_2 > 0$, is essentially the same.

Suppose that

$$(\lambda - A(q))(u, v, \theta) = (\varphi, \psi, \eta), \qquad (3.10)$$

and

$$(\lambda - A^n(q^n))(u^{n_1}, v^{n_1}, \theta^{n_2}) = P^n(\varphi, \psi, \eta)$$
$$= (P_1^{n_1} \varphi, P_1^{n_1} \psi, P_2^{n_2} \eta). \qquad (3.11)$$

We shall show that $(u^{n_1}, v^{n_1}, \theta^{n_2}) \longrightarrow (u, v, \theta)$ as $n \to \infty$. Recall the proof of Theorem 2.1 and let H^n denote the finite dimensional subspace of H given by $H^n = H_1^{n_1} \times H_2^{n_2}$. Note that $H^n \subset V$ for all $n = (n_1, n_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+$. For $\lambda > 0, n = (n_1, n_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+$, and $q \in Q$, define $A_\lambda^n(q) \in \mathcal{L}(H^n)$ by

$$A_\lambda^n(q) = \begin{bmatrix} \lambda^2 + \lambda C^{n_1}(q) + A_1^{n_1}(q) & L^n(q)^* \\ -L^n(q) & \left(1 + \lambda^{-1} A_2^{n_2}(q)\right) \end{bmatrix}$$

It follows that

$$\langle A_\lambda^n(q) \Phi^n, \Psi^n \rangle = \langle A_\lambda(q) \Phi^n, \Psi^n \rangle \qquad (3.12)$$

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for $\Phi^n = (\Phi_1^{n_1}, \Phi_2^{n_2})$, $\Psi = (\Psi_1^{n_1}, \Psi_2^{n_2}) \in H^n$, where the inner product $\langle \cdot, \cdot \rangle$ on H is given by (2.13) and the operator $A_\lambda(q) \in \mathcal{L}(V, V^*)$ is given by (2.15). From (2.16) and (3.12) we obtain that

$$\langle A_\lambda^n(q)\Phi^n, \Phi^n \rangle \geq \beta_\lambda \|\Phi^n\|^2, \qquad \Phi^n \in H^n \tag{3.13}$$

where $\beta_\lambda = \min(\beta_1, \lambda^- \beta_2)$ and $\|\cdot\|$ is the norm given in (2.14).

From (3.10) and (3.11), as was the case in the proof of Theorem 2.1 (see (2.11) and (2.12)), we find that

$$A_\lambda(q)(u, \theta) = \Phi = (\Phi_1, \Phi_2),$$

$$v = \lambda u - \varphi, \tag{3.14}$$

and

$$A_\lambda^n(q^n)(u^{n_1}, \theta^{n_2}) = \Phi^n = (\Phi_1^{n_1}, \Phi_2^{n_2}),$$

$$v^{n_1} = \lambda u^{n_1} - P_1^{n_1}\varphi, \tag{3.15}$$

where $A_\lambda(q)$ is given by (2.15), $\Phi^n = (\Phi_1^{n_1}, \Phi_2^{n_2}) \in H^n$ is given by

$$\Phi_1^{n_1} = P_1^{n_1}\psi + (C^{n_1}(q^n) + \lambda) P_1^{n_1}\varphi,$$

$$\Phi_2^{n_2} = \lambda^{-1} (P_2^{n_2}\eta + L^n(q^n) P_1^{n_1}\varphi),$$

and $\Phi = (\Phi_1, \Phi_2) \in V^*$ is given by

$$\Phi_1 = \psi + (C(q) + \lambda)\varphi,$$

$$\Phi_2 = \lambda^{-1}(\eta + L(q)\varphi).$$

Using assumptions (C), (E), and (G), it is not difficult to argue that the following assertions hold.

(i)

$$\lim_{n \to \infty} \|\Phi^n - \Phi\|_* = 0,$$

where $\|\cdot\|_*$ denotes the usual operator norm on V^*.

(ii) The map $q \to A_\lambda(q)\Psi$ is continuous from $Q \subset \mathcal{Q}$ into V^* for each $\Psi \in V$, and in particular

$$\lim_{n \to \infty} \|A_\lambda(q^n)\Psi - A_\lambda(q)\Psi\|_* = 0, \qquad \Psi \in V$$

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whenever $q^n \to q$.

(iii) There exists a positive constant ρ_λ, independent of $q \in Q$, for which

$$\|A_\lambda(q)\Psi\|_* \le \rho_\lambda \|\Psi\|, \qquad \Psi \in V.$$

Furthermore, if we let $P^n : H \to H^n$ denote the orthogonal projection of H onto H^n with respect to the inner product $\langle \cdot, \cdot \rangle$ given by (2.13), then $P^n \Psi = (P_1^{n_1} \Psi_1, P_2^{n_2} \Psi_2), \Psi = (\Psi_1, \Psi_2) \in H$. It follows from assumption (G) that

$$\lim_{n \to \infty} \|P^n \Psi - \Psi\| = 0, \qquad \Psi \in V, \tag{3.16}$$

and

$$\lim_{n \to \infty} |P^n \Psi - \Psi| = 0, \qquad \Psi \in H.$$

Setting $\Psi = (u, \theta)$ and $\Psi^n = (u^{n_1}, \theta^{n_2})$, from (3.13) and (iii) above, we obtain

$$\beta_\lambda \|\Psi^n - P^n \Psi\|^2 \le \langle A_\lambda^n(q^n)(\Psi^n - P^n \Psi), \Psi^n - P^n \Psi \rangle$$

$$= \langle A_\lambda^n(q^n)\Psi^n - A_\lambda(q)\Psi, \Psi^n - P^n \Psi \rangle$$

$$+ \langle A_\lambda(q)\Psi - A_\lambda(q^n)\Psi, \Psi^n - P^n \Psi \rangle$$

$$+ \langle A_\lambda(q^n)\Psi - A_\lambda(q^n)P^n \Psi, \Psi^n - P^n \Psi \rangle$$

$$\le \|\Phi^n - \Phi\|_* \|\Psi^n - P^n \Psi\| + \|A_\lambda(q)\Psi - A_\lambda(q^n)\Psi\|_*$$

$$\times \|\Psi^n - P^n \Psi\| + \|A_\lambda(q^n)(\Psi - P^n \Psi)\|_* \|\Psi^n - P^n \Psi\|$$

$$\le \frac{3}{2\beta_\lambda}\left\{ \|\Phi^n - \Phi\|_*^2 + \|A_\lambda(q)\Psi - A_\lambda(q^n)\Psi\|_*^2 + \rho_\lambda^2 \|\Psi - P^n \Psi\|^2 \right\}$$

$$+ \frac{\beta_\lambda}{2} \|\Psi^n - P^n \Psi\|^2.$$

It follows therefore, from (i), (iii) and (3.16) above, that

$$\lim_{n \to \infty} \|\Psi^n - P^n \Psi\|^2 \le \frac{3}{\beta_\lambda^2} \lim_{n \to \infty}\left\{ \|\Phi^n - \Phi\|_*^2 + \|A_\lambda(q)\Psi - A_\lambda(q^n)\Psi\|_*^2 \right.$$

$$\left. + \rho_\lambda^2 \|\Psi^n - P^n \Psi\|^2 \right\} = 0. \tag{3.17}$$

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The estimate (3.17) together with (3.14), (3.15), (3.16) and assumption (G) yield that

$$\lim_{n \to \infty} (u^{n_1}, v^{n_1}, \theta^{n_2}) = (u, v, \theta)$$

in X and the convergence asserted in (3.9) has been established.

Remark   We note that if the admissible parameter set Q is infinite dimensional (i.e. functional), its discretization could also be included within our general approximation framework and convergence theory, as well. For a detailed description of how this can be accomplished, see for example, [3].

4. Examples and Numerical Results

We consider the basic equations of one dimensional linear thermoelasticity (see [5] [16])

$$\rho \frac{\partial^2 u}{\partial t^2} - \alpha_V (\lambda + 2\mu) \frac{\partial^3 u}{\partial \eta^2 \partial t^2} - (\lambda + 2\mu) \frac{\partial^2 u}{\partial \eta^2} \\ + \alpha_T (3\lambda + 2\mu) \frac{\partial \theta}{\partial \eta} = f_0, \qquad 0 < \eta < \ell, \quad t > 0 \tag{4.1}$$

$$\rho c \frac{\partial \theta}{\partial t} - \kappa \frac{\partial^2 \theta}{\partial \eta^2} + \bar{\theta} \alpha_T (3\lambda + 2\mu) \frac{\partial^3 u}{\partial \eta^2 \partial t} = g_0, \tag{4.2}$$

$$0 < \eta < \ell, \quad t > 0.$$

With the introduction into the first equation above of the Voigt-Kelvin viscoelastic damping term, equations (4.1) and (4.2) describe the longitudinal, or axial, vibrations of a thin visco-thermoelastic rod of length ℓ. In the above equations, u denotes the axial displacement, θ the absolute temperature, ρ is the mass density of the rod, λ and μ are the Lamé parameters, c is the rod's specific heat and κ is its thermal conductivity. The positive constant $\bar{\theta}$ is known as the reference temperature - the absolute temperature of a stress free reference state for the rod. The functions f_0 and g_0 represent, respectively, an externally applied axial force and thermal input. The nonnegative constants α_V and α_T denote the viscosity coefficient and the coefficient of thermal expansion, respectively.

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We are interested in studying the partial differential equations (4.1), (4.2) together with initial conditions of the form

$$u(0,\eta) = u_0(\eta), \frac{\partial u}{\partial t}(0,\eta) = v_0(\eta), \theta(0,\eta) = \theta_0(\eta), \tag{4.3}$$

for $0 < \eta < \ell$, and one or the other of the following two sets of boundary conditions.

(I) Clamped ends with temperature fixed –

$$u(t,0) = u(t,\ell) = 0, \qquad t > 0$$

$$\theta(t,0) = \theta(t,\ell) = 0.$$

(II) Clamped and insulated ends –

$$u(t,0) = u(t,\ell) = 0, \qquad t > 0$$

$$\frac{\partial \theta}{\partial \eta}(t,0) = \frac{\partial \theta}{\partial \eta}(t,\ell) = 0, \qquad t > 0.$$

We note that our theory can of course handle a variety of other boundary conditions. For our purposes here, however, to simply illustrate the application of our general appraoch, the two given in (I) and (II) above will suffice.

We consider the problem of identifying some subset of the parameters

$$q = (\alpha_V, \alpha_T, \lambda, \mu, \kappa) \tag{4.4}$$

over some closed and bounded subset Q contained in the positive orthant of the Euclidean space $Q = \mathbf{R}^5$. Once again we note that our theory, or some relatively minor modification of it, can handle inverse problems involving the estimation of any of the parameters, input functions, or initial data appearing in either the differential equations (4.1), (4.2) or initial conditions (4.3). We consider only the subset given in (4.4) simply for the purpose of illustration.

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In order to apply the abstract theory developed in the second and third sections above, we make the following definitions, identifications and assumptions. We set $H_1 = L_2(0, \ell)$ endowed with the standard L_2 inner product (to be denoted by $\langle \cdot, \cdot \rangle_1$) and set $H_1 = L_2(0, \ell)$ with the weighted inner product

$$\langle \varphi, \psi \rangle_2 = \int_0^\ell \frac{c}{\theta} \varphi \psi.$$

In the case of the boundary conditions (I) we define both V_1 and V_2 to be the Sobolev space $H_0^1(0, \ell)$ with the boundary conditions (II) we take $V_1 = H_0^1(0, \ell)$ and $V_2 = H^1(0, \ell)$. The Sobolev spaces $H_0^1(0, \ell)$ and $H^1(0, \ell)$ are assumed to be endowed with the usual Sobolev inner products and induced Sobolev norms.

For each $q \in Q$ we define the operators $A_j(q) \in \mathcal{L}(V_j, V_j^*), j = 1, 2$, $C(q) \in \mathcal{L}(V_1, V_1^*)$, and $L(q) \in \mathcal{L}(V_2, V_2^*)$ by

$$(A_1(q)\varphi)(\psi) = \int_0^\ell \frac{\lambda + 2\mu}{\rho} D\varphi D\psi, \qquad \varphi, \psi \in V_1, \tag{4.5}$$

$$(A_2(q)\varphi)(\psi) = \int_0^\ell \frac{\kappa}{\rho c} D\varphi D\psi, \qquad \varphi, \psi \in V_2, \tag{4.6}$$

$$(C(q)\varphi)(\psi) = \int_0^\ell \alpha_V \frac{(\lambda + 2\mu)}{\rho} D\varphi D\psi, \qquad \varphi, \psi \in V_1, \tag{4.7}$$

and

$$(L(q)\varphi)(\psi) = -\int_0^\ell \alpha_T \frac{\bar{\theta}(3\lambda + 2\mu)}{\rho c} D\varphi \psi, \qquad \varphi \in V_1, \psi \in V_2. \tag{4.8}$$

Definition (2.4) then yields that the operator $L(q)^* \in \mathcal{L}(V_2, V_1^*)$ is given by

$$(L(q)^* \psi)(\varphi) = \int_0^\ell \frac{\alpha_T(3\lambda + 2\mu)}{\rho} D\psi \varphi, \qquad \varphi \in V_1, \psi \in V_2. \tag{4.9}$$

We assume that $u_0 \in H_0^1(0, \ell)$, $v_0, \theta_0 \in L_2(0, \ell)$, $f_0, g_0 \in L_1(0, T, L_i(0, \ell))$ and set $f(t, \eta) = \frac{1}{\rho} f_0(t, \eta)$, $g(t, \eta) = \frac{1}{\rho c} g_0(t, \eta)$ for almost every $(t, \eta) \in [0, T] \times [0, \ell]$. It is then a simple and straight forward matter to show that the assumptions (A) - (F) stated in Section 2 are satisfied.

For each $N = 1, 2, \ldots$ let $\left\{ \varphi_j^N \right\}_{j=0}^N$ denote the standard linear B-spline (i.e. "pup-tent" or "hat") functions on the interval $[0, \ell]$ defined with respect to the uniform mesh, $\{0, \ell/N, 2\ell/N, \ldots, \ell\}$. If for $n_1 = 2, 3, \ldots$ we set $H_1^{n_1} = span \left\{ \varphi_j^{n_1} \right\}_{j=1}^{n_1-1}$ and for each $n_2 = 2, 3, \ldots$ we set $H_2^{n_2} = span \left\{ \varphi_j^{n_2} \right\}_{j=1}^{n_2-1}$ in the case of boundary conditions (I) or $H_2^{n_2} = span \left\{ \varphi_j^{n_2} \right\}_{j=0}^{n_2}$ in the case of boundary conditions (II), then $H_j^{n_j} \subset V_j$, $j = 1, 2$, and , using standard approximation estimates for interpolatory splines (see [14]) together with the Schmidt inequality (see [11]) it is not difficult to argue that assumption (G) is satisfied (see, for example, [10], [11]).

With the spline bases for $H_1^{n_1}$ and $H_2^{n_2}$ defined in the previous paragraph and using the definitions (4.5) - (4.9), it is straight forward to write the finite dimensional initial value problems (3.1) - (3.3) (or equivalently, (3.4), (3.5)) in matrix form. In the examples to follow we took the observation space Z to be $L_2(0, \ell)$ and chose the functionals Φ_i as

$$\Phi_i(x, q; z) = \Phi_i((u, v, \theta), q; z) = \int_0^\ell |u - z|^2 ,$$
$$i = 1, 2, \ldots, \nu.$$

We used the IMSL routine $ZXSSQ$, an implementation of the iterative Newton's method/Steepest descent hybrid Levenberg-Marquardt algorithm to solve the resulting finite dimensional nonlinear least squares minimization problems (ID^n). The IMSL routine IVPAG, an implementation of Gear's method for stiff systems, was used in each iteration to solve the initial value problems (3.4), (3.5) for a given choice of $q \in Q$, whenever required. A composite two-point Gauss Legendre quadrature formula was used to numerically compute integrals whenever necessary (in inner products when computing orthogonal projections, for example). All codes were written in Fortran and all computations were carried out on the Cray XMP/48 at the San Diego Supercomputer Center. We note however, that since we were only interested in estimating constant (as opposed to functional) parameters, the use of the supercomputer was probably superfluous.

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Example 4.1 In this example we seek to 1) simply illustrate that our general approach is valid in practice as well as in theory and that the spline based scheme that has been proposed above performs satisfactorily, and 2) to obtain some feel for how good of a fit can actually be obtained. This second goal will aid us to a certain extent in interpreting our findings in Example 4.2 to follow where we consider a somewhat more realistic inverse problem.

In order to achieve the two above stated goals the following procedure was used. So called "true" values for the unknown parameters and functions u and θ were chosen. Then f_0 and g_0 were determined so that the partial differential equations (4.1) and (4.2) were satisfied and u_0, v_0, and θ_0 were chosen in accordance with (4.3). In this example we considered the boundary conditions specified in (I).

We chose

$$\rho = \alpha_V = \lambda = \mu = \alpha_T = c = \kappa = \bar{\theta} = \ell = 1,$$

$$u(t, \eta) = (\sin t)(\sin \pi \eta), \qquad 0 < \eta < 1, \quad t \geq 0,$$

and

$$\theta(t, \eta) = e^t \left(\eta^2 - \eta\right), \quad 0 < \eta < 1, \quad t \geq 0.$$

Observations were generated according to

$$z(t_i, \eta) = (\sin t_i)(\sin \pi \eta), \qquad 0 < \eta < 1,$$

for $t_i = .5i, i = 1, 2, \ldots, 8 = \nu$. The discretization levels n_1 and n_2 in our spline scheme were always chosen the same. Thus, without fear of confusion, we shall refer to their value as $n = n_1 = n_2$. Note that in this case, the dimension of the approximating system of ordinary differential equations (3.4), (3.5) is $3n - 3$

(a) In this example we seek to identify $q = (\alpha_V, \alpha_T)$. As an initial guess with which to start the iterative optimizing search we used $q^0 = (.25, .25)$. Our results are summarized in Table 4.1a below.

n	$\bar\alpha_V^n$	$\bar\alpha_T^n$	$J^n(q^0)$	$J^n(\bar q^n)$	CPU(m:s)
4	.99045	.62424	3.41	5.13×10^{-1}	0:6.34
8	1.0159	.90622	3.73	2.97×10^{-2}	1:46.17
12	1.0025	.98411	3.88	5.74×10^{-3}	5:46.50
16	1.0020	1.0037	3.94	2.36×10^{-3}	11:45.45

Table 4.1a

(b) In this example we consider a somewhat more challenging problem; we seek to identify the five

parameters $q = (\alpha_V, \alpha_T, \lambda, \mu, \kappa)$. As an initial guess we used $q^0 = (.25, .25, .25, .25, .25)$. Our

results are displayed in Table 4.1b below

n	$\bar\alpha_V^n$	$\bar\alpha_T^n$	$\bar\lambda^n$	$\bar\mu^n$	$\bar\kappa^n$	$J^n(q^0)$	$J^n(\bar q^n)$	CPU(m:s)
4	.89827	1.0720	.99102	1.0393	.93712	8.25×10^2	2.24×10^{-1}	1:18.14
8	1.0090	1.0102	.94168	.98898	.98229	9.17×10^2	1.86×10^{-2}	4:56.44
12	.99566	.99349	.98076	1.0016	.99094	9.34×10^2	4.41×10^{-3}	27:42.44
16	1.0028	1.0072	.97233	1.0058	.99443	9.41×10^2	1.32×10^{-3}	40:19.79

Table 4.1b

<u>Example 4.2</u> In this series of tests we indicate how our approach might be applied in practice,

and how it might perform. In particular we illustrate how it may serve as an aid in the identification

of internal dissipation (i.e. damping) mechanisms in flexible structures.

We consider a long, slender aluminum (AL T2024 - T4) rod of length 100 in. which is clamped

and insulated at both ends (i.e. boundary conditions (II)). We assume that it is set to vibrating

via a mechanical input of the form

$$f_0(t,\eta) = \begin{cases} a_0 \chi_{[\epsilon_1, \epsilon_2]}(\eta) \sin 3\pi t, & 0 < \eta < \ell, \quad 0 \le t \le T_0 \\ 0, & T_0 < t \le T, \end{cases}$$

where χ_I denotes the characteristic function for the interval $I \subset \mathbf{R}$. We assume that there is no

externally applied thermal input – i.e. we take

$$g_0(t,\eta) = 0, \quad 0 < \eta < \ell, t \ge 0,$$

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and we assume that the system is initially at rest. That is, we take $u_0 = v_0 = 0$, and $\theta_0 = \dot{\theta}, 0 < \eta < \ell$.

After appropriate normalization, the laboratory or experimentally measured values for the material parameters appearing in the system (4.1), (4.2) are given by

$$
\begin{aligned}
\ell &= 1 \\
\rho &= 9.82 \times 10^{-2} & \alpha_T &= 1.29 \times 10^{-3} \\
\lambda &= 2.0635 \times 10^{-1} & C &= 5.39611 \times 10^{-1} \\
\mu &= 1.1111 \times 10^{-1} & \kappa &= 7.02176 \times 10^{-7}.
\end{aligned}
\tag{4.10}
$$

There is no generally accepted means to explicity measure the viscosity coefficient α_V in the laboratory. Typically its value is determined via an identification procedure such as the one which we have developed here. We took $\hat{\theta} = 6.8 \times 10^1$.

With the boundary conditions given in (II), the natural mode shapes for the homogeneous system corresponding to (4.1), (4.2) are of the form $\Phi_0 = (u_0, \theta_0) = (0, 1)$, $\Phi_{m,j} = (u_m, \theta_{m,j}) = \left(\sin \frac{m\pi\eta}{\ell}, c_{m,j} \cos \frac{m\pi\eta}{\ell}\right)$, $m = 1, 2, \ldots, j = 1, 2, 3$, where for each $m = 1, 2, \ldots$, and $j = 1, 2, 3$, $c_{m,j}$ is a complex constant. Thus when a Galerkin scheme using $M + 1$ elements of the form $\Psi_m = \left(\sin \frac{m\pi\eta}{\ell}, \cos \frac{m\pi\eta}{\ell}\right)$, $m = 0, 1, 2, \ldots, M$, is employed to discretize (4.1), (4.2), the resulting finite dimensional equivalent first order system consists of $M + 1$ decoupled first order systems, one of dimension one and the remaining M each of dimension three. We used such a scheme with the true values of the parameters and $M = 24$ to generate simulated observational data upon which to base our fits. Once again in our spline scheme we use $n = n_1 = n_2$ to denote the level of discretization.

The presence of either, and certainly both, the Voigt-Kelvin viscoelastic term (i.e. $\alpha_V > 0$) and the thermoelastic term (i.e. $\alpha_T > 0$) yield some form of mechanical energy dissipation. In fact, when the boundary conditions are as given in (II), with a simple change of variable and the construction of an appropriate Lyapunov functional (or, equivalent norm), it can be shown that either the Voigt-Kelvin damping or the thermoelastic coupling leads to a uniformly exponentially stable open-loop or unforced system (i.e. $\left|\tilde{S}(t; q)\right|_X \leq \mathcal{M} \exp(\omega t), t \geq 0$, with $\omega < 0$, where

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$\left\{ \tilde{S}(t;q) : t \geq 0 \right\}$ is the semigroup corresponding to the transformed system (see [6] and [7]). The simple change of variable consists of transforming the absolute temperature θ to the temperature variation from the time invariant average temperature over the length of the rod, θ_{AVE}. That such a constant average temperature exists, independent of the input f_0 and initial conditions (4.3), is a direct consequence of the insulated ends. Indeed, integrating equation (4.2) from 0 to ℓ, integrating by parts and then imposing the boundary conditions (II), yields the desired conclusion. We note that even without the change of variable, the mechanical vibrations of the unforced system still tend to zero exponentialy fast. (In this case the absolute temperature tends to θ_{AVE} exponentialy fast.) Thus the presence of either the viscoelastic (Voigt-Kelvin) or the thermoelastic effects will exponentially damp unforced mechanical vibrations. In the series of examples to follow we illustrate how our scheme and general approach might be used to aid in the identification of the mechanism for, and/or an appropriate model for an observed dissipation of mechanical energy in a flexible structure.

(a) In this test we took $a_0 = 1, \varepsilon_1 = .4$, $\varepsilon_2 = .435$, $T_0 = .5$ and $T = 5$. We set $\alpha_V = 0$ (i.e. no Voigt-Kelvin damping), generated observations at times $t_i = .25i, i = 0, 1, 2, \ldots, 20$ via the modal scheme described above, and used our scheme to see if we could estimate α_T. All other parameters were taken as given in (4.10) and α_V was held fixed at zero. For an initial guess we set $\alpha_T^0 = 10^{-2}$. Our results are summarized in Table 4.2a below. It is clear from the table that n had to be taken quite large in order to obtain an accurate estimate for α_T. It is likely that the reason for this is that thermoelastic effects in metals are relatively slight. Consequently we would expect that the thermo-mechanical coupling as described by the parameter α_T is not readily discernable by our approximation and identification procedures.

n	$\bar{\alpha}_T^n$	$J^n(q^0)$	$J^n(\bar{q}^n)$	CPU(m:s)
4	3.0409×10^{-3}	1.189×10^{-1}	1.187×10^{-1}	0:4.94
8	4.4408×10^{-3}	8.667×10^{-2}	8.369×10^{-2}	1:11.12
12	2.3034×10^{-3}	5.556×10^{-2}	5.345×10^{-2}	3:15.86
16	1.8457×10^{-3}	4.99×10^{-2}	4.659×10^{-2}	21:32.26
20	1.2897×10^{-3}	4.471×10^{-2}	3.918×10^{-2}	45:36.64

Table 4.2a

(b) This test is similar to the previous one described in (a) above, however, this time we introduced some Voigt-Kelvin damping into the system. We set $\alpha_V = 10^{-2}$ when the observations were generated. This time we also took $a_0 = 5.0, T = 3.0$, and once again set $\alpha_T^0 = 10^{-2}$. Our findings are reported in Table 4.2b. In general, the presence of the Voigt-Kelvin damping tended to make our task somewhat simpler and to improve the overall performance of our scheme. This is not surprising since Voigt-Kelvin is a rather strong form of dissipation and its presence would be likely to significantly increase the stability and accuracy of our numerical integrator.

n	$\bar{\alpha}_T^n$	$J^n(q^0)$	$J^n(q^n)$	CPU(m:s)
4	2.1998×10^{-4}	1.607×10^{-1}	1.570×10^{-1}	0:54.21
8	5.5681×10^{-4}	6.745×10^{-2}	6.376×10^{-2}	1:3.82
16	1.2749×10^{-3}	1.453×10^{-2}	9.691×10^{-3}	4:35.93
20	1.2888×10^{-3}	1.765×10^{-2}	1.361×10^{-2}	14:17.86

Table 4.2b

(c) In this example we first attempt to fit a model with only thermoelastic damping to observations of a system which has both visco- and theoremoelastic dissipation. We then try to fit a model with viscoelastic damping only to a system in which the dissipation is due only to thermal effects. In both tests the input excitation and observations are as they were in test (b) above.

For the first test we generated observations with $\alpha_V = 10^{-2}$ and α_T as given in (4.10). We then attempted to estimate the parameter α_V with α_T held fixed at zero. The initial guess for α_V was taken to be $\alpha_V^0 = 2.5 \times 10^{-1}$. Our results are given in Table 4.2c.1.

In the second test, observations were generated with $\alpha_V = 0$ and α_T as given in (4.10). We

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then again attempted to estimate α_V with α_T fixed at zero. The initial guess was $\alpha_V^0 = 10^{-2}$. The results for this test are given in Table 4.2c.2

n	$\bar{\alpha}_V^n$	$J^n(q^0)$	$J^n(\bar{q}^n)$	CPU(m:s)
4	7.1179x10^{-3}	4.234x10^{-1}	1.466x10^{-1}	0:10.23
8	1.2178x10^{-2}	4.122x10^{-1}	4.863x10^{-2}	0:44.62
16	9.7023x10^{-3}	4.157x10^{-1}	1.064x10^{-2}	9:12.97
20	9.7812x10^{-3}	4.165x10^{-1}	1.135x10^{-2}	16:17.10

Table 4.2c.1

n	$\bar{\alpha}_V^n$	$J^n(q^0)$	$J^n(\bar{q}^n)$	CPU(m:s)
4	1.4501x10^{-3}	3.329x10^{-1}	3.019x10^{-1}	0:8.68
8	2.2659x10^{-3}	1.951x10^{-1}	1.294x10^{-1}	0:24.14
16	2.1950x10^{-4}	2.096x10^{-1}	4.695x10^{-2}	3:43.28
20	4.9641x10^{-4}	2.168x10^{-1}	4.469x10^{-2}	12:53.48

Table 4.2c.2

For the system that we have considered here, the results given in Tables 4.2c.1 and 4.2c.2 seem to contribute to the conclusion that the effects of thermoelastic damping are insignificant. Indeed, in the first test, a relatively accurate estimate for the viscosity coefficient α_V can be obtained with a model that ignores all thermoelastic effects. The second test, although not conclusively, appeared to indicate that the presence of thermoelastic effects in the data will have only a minimal effect on the estimates obtained for the viscosity term α_V

We note at this point however, with a word of caution, that the interpretation given in the previous paragraph is not intended as a sweeping generalization and conclusion about thermoelastic effects in the vibration of flexible structures. Indeed, we present these findings simply as an illustration of how our general approach and schemes could be used in such a damping study. To draw such conclusions would necessarily require an exhaustive investigation involving both simulation and computational studies as well as laboratory experimentation.

5. Summary and Concluding Remarks

In this paper we have considered inverse or parameter identification problems for thermoelastic systems. We defined a class of abstract infinite dimensional systems consisting of a second order elasticity equation coupled with a first order abstract parabolic equation. We established the well-posedness of this system by rewriting it as an equivalent abstract first order system set in an appropriate product Hilbert space and then applying results from linear semigroup theory. A class of identification problems in which the state constraints were given by the mild solution to the abstract thermoelastic system were defined. We then proceeded to develop an abstract approximation framework based upon generic Galerkin approximation of the mild solutions. Using approximation results from linear semigroup theory, we were, under appropriate assumptions, able to establish that the solutions to the resulting finite dimensional approximating parameter estimation problems, in some sense, approximate a solution to the original infinte dimensional identification problem. We demonstrated that our approach could be applied to the basic equations of linear thermoelasticity and provided computational results as evidence that our schemes are not only theoretically well founded, but numerically sound and pactical as well.

Many interesting and important open questions related to the identification and control of abstract thermoelastic systems remain. In particular, nonlinear systems and especially ones in which the material parameters are permitted to be temperature dependent need to be considered. Also, numerical studies using actual experimental data rather than simulation data should be carried out and reported on. With regard to the optimal control problem, a complete understanding of the asymptotic behavior of the spectrum of the operator $A(q)$ given by (2.6) would be extremely useful as would it be for just the basic equations of linear themoelasticity (4.1), (4.2). Indeed, for the boundary equations (II) it can be shown that the eigenvalues asymptotically approach a straight line in the left half plane. For other standard sets of boundary conditions, in particular the Dirichlet/Dirichlet case given in (I), this author knows of no similar types of theoretical results.

Related to this is the question of uniform exponential stability of the open-loop semigroups. For some sets of boundary conditions, this can be established (see [7]). However, the question in general is still open. Finally, when doing approximation for LQG control, among the conditions required to establish convergence of the optimal closed-loop feedback gains, is the preservation of uniform exponential stability under approximation (see, for example, [6]). Even for a linear spline based Galerkin scheme for the basic equations of linear thermoelasticity with the boundary conditions given in (II), this condition does not appear to hold. Schemes for which this condition can be verified (other than modal schemes of course, for which it is trivially true) need to be developed and studied.

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16. Abstract

An abstract approximation framework and convergence theory for the identification of thermoelastic systems is developed. Starting from an abstract operator formulation consisting of a coupled second order hyperbolic equation of elasticity and first order parabolic equation for heat conduction, well-posedness is established using linear semigroup theory in Hilbert space, and a class of parameter estimation problems is then defined involving mild solutions. The approximation framework is based upon generic Galerkin approximation of the mild solutions, and convergence of solutions of the resulting sequence af approximating finite dimensional parameter identification problems to a solution of the original infinite dimensional inverse problem is established using approximation results for operator semigroups. An example involving the basic equations of one dimensional linear thermoelasticity and a linear spline based scheme is discussed and numerical results indication how our approach might be used in a study of damping mechanisms in flexible structures are presented.

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